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Continuous Dependence in Rational Chebyshev Approximation

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The dependence of best Chebyshev approximation by generalized abstract rational functions on the function being approximated is studied.

Let $X \neq \phi$ be a compact topological space and

$$\|g\| = \max\{|g(x)|: x \in X\}.$$

Let $\{\phi_1, ..., \phi_n\}$, $\{\psi_1, ..., \psi_m\}$ be linearly independent subsets of $C(X)$. Define

$$R(A, x) = P(A, x)/Q(A, x) = \sum_{k=1}^{n} a_k \phi_k(x) \Big/ \sum_{k=1}^{m} a_{n+k} \psi_k(x).$$

The conventions of Boehm (assuming his dense nonzero property is satisfied) or of Goldstein (stabilized rational functions) [14, pp. 84–89] can be used to give $R(A, x)$ a value when $Q(A, x) = 0$. Let K be a subset of $(n+m)$-space. The problem of K-rational approximation is given $f \in C(X)$ to choose A^* minimizing $e(A) = \|f - R(A, \cdot)\|$ over K. Such an A^* is called best and $R(A^*, \cdot)$ is called a best approximation to f. Denote A^* by $C(f)$ and $R(A^*, \cdot)$ by $T(f)$.

If K is closed and nonempty, a best coefficient and approximation exists by generalizations of the arguments of Boehm and Goldstein [Dunham, 9, 11].

$\{f_k\}$ will denote any sequence with limit f.

Now $R(\alpha A, x) = R(A, x)$ for $\alpha > 0$. Hence for convenience in existence and convergence arguments, we normalize rational functions $R(A, \cdot)$ so that

$$\sum_{k=1}^{m} |a_{n+k}| = 1. \tag{1}$$

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Let

$$K_{GE} = \{A: Q(A, \cdot) \geqslant 0, Q(A, \cdot) \not\equiv 0\},$$
$$K_0 = \{A: Q(A, \cdot) \not\equiv 0\}.$$

Under the normalization (1), K_{GE} and K_0 are closed.

THEOREM 1. *Let $\|R(A, \cdot)\| < \infty$ for some $A \in K$. Let K be closed. The sequence $C(f_k)$ has a limit point A. For any such limit point A, $R(A, \cdot)$ is a best approximation to f.*

Proof. Define

$$\|A\| = \sum_{k=1}^{n} |a_k|.$$

Let $C(f_k) = A^k$. Suppose $\{\|A^k\|\}$ is an unbounded sequence. By taking a subsequence if necessary, we can assume $\|A^k\| > k$. Define $B^k = A^k/\|A^k\|$. Then $\|B^k\| = 1$ and $\{B^k\}$ is a bounded sequence with accumulation point B. Assume without loss of generality that $\{B^k\} \to B$. Select $z \in X$ such that $|P(B, z)| > \varepsilon$. Then for all k sufficiently large $|P(B^k, z)| > \varepsilon$ and $P(A^k, z) > k\varepsilon$. As

$$|Q(A, z)| \leqslant \sum_{k=1}^{m} |\psi_k(z)|$$

for A satisfying (1), $|R(A^k, z)| \to \infty$ and $\|f_k - R(A^k, \cdot)\| \to \infty$. This gives a contradiction of A^k being best and so $\|A^k\|$ is bounded.

$\{A^k\}$ has a convergent subsequence, which we assume without loss of generality to be $\{A^k\}$, with limit A. We claim $R(A, \cdot)$ is best to f. Suppose not, then there exists a point x and $\varepsilon > 0$ such that

$$|f(x) - R(A, x)| > \|f - T(f)\| + \varepsilon. \tag{2}$$

The first possibility is that $Q(A, x) \neq 0$ and $P(A, x) = 0$. But in this case $R(A^k, x) \to \infty$, which is impossible. The second possibility is that $P(A, x) = Q(A, x) = 0$. In a Goldstein-type theory, $R(A, x)$ can always be defined equal to $f(x)$. In a Boehm-type theory, we can find in a neighbourhood of x a point at which $Q(A, \cdot)$ does not vanish and for which an inequality of the type (2) holds. Thus we need only consider the remaining possibility, which is that $Q(A, x) \neq 0$. In this case $f(x) - R(A^k, x) \to f(x) - R(A, x)$ and for all k sufficiently large

$$|f(x) - R(A^k, x)| > \|f - T(f)\| + \varepsilon,$$

hence

$$|f_k(x) - R(A^k, x)| > \|f_k - T(f)\| + (\varepsilon/2)$$

for all k sufficiently large. This contradicts optimality of A^k, hence (2) cannot hold, proving the theorem.

If $\{A^k\} \to A$ and $Q(A, \cdot)$ has no zeros, $\{R(A^k, \cdot)\}$ converges uniformly to $R(A, \cdot)$. An examination of the previous proof gives

COROLLARY 1. *Suppose f has a unique best approximation $R(A, \cdot)$, with $Q(A, \cdot) > 0$ and $R(A, \cdot)$ having a unique representation under* (1). *Then $C(f_k) \to A$ and $T(f_k)$ converges uniformly to $R(A, \cdot)$ on X.*

Remark. In case $K = K_{GE}$, the uniqueness and unique representation hypotheses are satisfied if the tangent space $S(A)$ of $R(A, \cdot)$ is a Haar subspace of dimension $n + m - 1$. For the arguments see Dunham [6].

In case $X = [\alpha, \beta]$ and we approximate by ordinary rationals, the above corollary and remark cannot be improved. The theory of Werner [17] shows that if $S(A)$ is of dimension $< n + m - 1$, discontinuity of T at f occurs if f is not an approximant.

Also of interest is $K_G = \{A: Q(A, \cdot) > 0\}$. K_G is not closed in general. Arguments of Dunham [6, Theorem 2] give

LEMMA. *Let $R(A, \cdot)$ be a unique best approximation from K_G and have a unique representation under* (1). *Then $R(A, \cdot)$ is a unique best approximation from K_{GE}.*

From Corollary 1 we get

THEOREM 2. *Let f have a unique best approximation $R(A, \cdot)$ from K_G with $R(A, \cdot)$ having a unique representation under* (1). *Then for all k sufficiently large, f_k has a best coefficient from K_G and $R(A^k, \cdot) \to R(A, \cdot)$ uniformly.*

The previously cited results of Werner show the necessity of unique representation for uniform convergence.

The case where X is not compact but f is bounded as well as continuous on X is also of interest. Applications include approximation by ordinary rationals on $[0, \infty)$ and $(-\infty, \infty)$. Existence is covered by the author in [11]. Theorem 1 holds. As convergence in coefficients may not imply uniform convergence even when $Q(A, \cdot) > 0$, Corollary 1 may not hold. Also the Haar condition does not necessarily imply uniqueness, hence the remark following the corollary does not hold. Examples of discontinuity of T are given by Blatt [2].

The case where all functions are complex-valued is also of interest, in which case we use K_0. Theorem 1 holds. Corollary 1 holds if X is compact, but may not hold if X is noncompact. There may be no global uniqueness result for rational complex Chebyshev approximation [Saff and Varga, 15, 16]. The real discontinuity result of Werner [17] and the result of Saff and Varga as to when real best approximations are complex best approximations show that nonuniform convergence of $T(f_k)$ to $T(f)$ can occur when functions are real-valued and X is a finite interval.

Let us consider an important case not covered exactly by our theory, the case of Chebyshev approximation with Hermite interpolation [Chalmers and Taylor, 4; Perrie, 13]. In this case K varies with f instead of being fixed. Assume coefficient vectors are selected from K_0 or K_{GE}. Under Boehm's convention, K is not necessarily closed [Dunham, 9, p. 286]. An extension of Goldstein's convention is to match derivatives of $R(A,x)$ with those of $f(x)$ if $P(A,x)=Q(A,x)=0$: with this convention, K is closed [Dunham, 11]. We henceforth assume this convention. We assume that $f_k^{(j)}(x) \to f^{(j)}(x)$ for all x and j in the interpolation.

Let us assume that $\{\|f_k - T(f_k)\|\}$ is bounded. The proof of Theorem 1 goes through except we need to prove that $R(A,\cdot)$ interpolates f. This is true at a point x if $Q(A,x) \neq 0$ since $R^{(j)}(A^k,x) \to R^{(j)}(A,x)$. If $Q(A,x)=0$ and $P(A,x) \neq 0$, $|R(A^k,x)| \to \infty$ and we have a contradiction. If $Q(A,x)=P(A,x)=0$, we can apply our extension of Goldstein's convention. Corollary 1 holds for compact X. Analogous of the Lemma and Theorem 2 hold for compact X. If X is noncompact, existence and Theorem 1 still hold. Positive remarks on complex approximation carry over.

References

1. H. P. Blatt, Rationale Tschebyscheff-approximationen über unbeschränkten Intervallen, *Numer. Math.* **27** (1977), 179–190.
2. H. P. Blatt, Zur Stetigkeit rationaler T-approximationen über unbeschränkten Intervallen, *Numer. Math.* **27** (1977), 191–202.
3. D. Brink and G. D. Taylor, Chebyshev approximation by reciprocals of polynomials on $[0,\infty)$, *J. Approx. Theory* **16** (1976), 142–149.
4. B. L. Chalmers and G. D. Taylor, Uniform approximation with constraints, *Jahresber. Dtsch. Math. Verein* **81** (1979), 49–86.
5. E. W. Cheney and H. L. Loeb, On the continuity of rational approximation operators, *Arch. Rat. Mech. Anal.* **21** (1966), 391–401.
6. C. B. Dunham, Rational Chebyshev approximation on subsets, *J. Approx. Theory* **1** (1968), 484–487.
7. C. B. Dunham, Existence of admissible rational Chebyshev approximations on subsets, *J. Approx. Theory* **13** (1974), 307–308.
8. C. B. Dunham, Chebyshev approximation by interpolating rationals on $[\alpha,\infty)$, *Math. Comp.* **29** (1975), 549–551.
9. C. B. Dunham, Existence of transformed rational complex Chebyshev approximations, *J. Approx. Theory* **20** (1977), 284–287.

10. C. B. DUNHAM, Dependence of best rational Chebyshev approximations on the domain, *Canad. Math. Bull.* **20** (1977), 451–454.
11. C. B. DUNHAM, Existence of transformed rational complex Chebyshev approximations, II, in preparation.
12. S. ELLACOTT AND J. WILLIAMS, Rational Chebyshev approximation in the complex plane, *SIAM J. Numer. Anal.* **13** (1976), 310–323.
13. A. L. PERRIE, Uniform rational approximation with osculatory interpolation, *J. Comput. System Sci.* **4** (1970), 509–522.
14. J. RICE, "The Approximation of Functions," Vol. 2, Addison–Wesley, Reading, Mass., 1969.
15. E. B. SAFF AND R. S. VARGA, Nonuniqueness of best approximating complex rational functions, *Bull. Amer. Math. Soc.* **83** (1977), 375–377.
16. E. B. SAFF AND R. S. VARGA, Nonuniqueness of best complex rational approximations to real functions on real intervals, *J. Approx. Theory* **23** (1978), 78–85.
17. H. WERNER, On the rational Tschebyscheff-operator, *Math. Z.* **86** (1964), 317–326.